GAP Laboratory Session
Some computations with finitely presented groups

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1 Tutorials

http://turnbull.mcs.st-and.ac.uk/circa/gapstuff/gapfiles/fpres.html
http://turnbull.mcs.st-and.ac.uk/circa/gapstuff/gapfiles/fpres2.html

2 More challenging problems

Investigate some of the following questions or some fp-groups of your choosing.

2.1 Mennicke groups

Mathematical problem: Prove that $\langle x, y, z \mid x^y x^{-3}, y^z y^{-2}, z^x z^{-4} \rangle$ is a group of order 210. What is its structure?

Computational problem: How well can you do the coset enumeration over $\langle x \rangle$? (Hint: needs access to share package ACE so not easy to do in a Windows environment.)

Generalization: Consider $\langle x, y, z \mid x^y x^{-k}, y^z y^{-m}, z^x z^{-n} \rangle$ for various k, m, n.

Specialization: What is the order of $\langle x, y, z \mid x^y x^{-2}, y^z y^{-2}, z^x z^{-2} \rangle$?

Theoretical problem (for after the lab session): Prove some general results about these groups (see Mennicke's 1959 paper).

J. Mennicke, Einige endliche Gruppe mit drei Erzeugenden und drei Relationen, *Arch. Math.* **X** (1959) pp. 409–418.
George Havas and Colin Ramsay, Proving a group trivial made easy: a case study in coset enumeration, *Bulletin of the Australian Mathematical Society* **62** (2000) 105–118.

2.2 Fibonacci groups

Introduced by John Conway in 1965, the Fibonacci group $F(2, n)$ is generated by $x_1, \ldots, x_n$ with defining relations $x_i x_{i+1} = x_{i+2}, i = 1, \ldots, n$, where the subscripts are taken modulo n.

Theoretical results show that these groups are infinite for $n > 10$. Identify $F(2,n)$ for $3 \leq n \leq 10$. (More information: $F(2,7)$ and $F(2,9)$ were first identified using computer calculations. Warning: $F(2,9)$ is hard.)

J.H. Conway. Problem #5327. *American Mathematical Monthly*, 72:915, 1965. Solutions, by Conway et al., given in: Generators and relations for cyclic groups, *Ibid.*, 74:91–93, 1967.
George Havas. Computer aided determination of a Fibonacci group. *Bulletin of the Australian Mathematical Society*, 15:297–305, 1976.
D.F. Holt, An alternative proof that the Fibonacci group $F(2,9)$ is infinite, *Experimental Mathematics* **4** (1995) 97–100.
M.F. Newman, Proving a group infinite, *Arch. Math.* **54** (1990) 209–211.

2.3 Perfect groups

The following presentations arose in a census of perfect groups. Identify the groups. (We adopt the convention of using upper-case letters to denote inverses so that, for example, $A = a^{-1}$, etc.)

$\langle a, b \mid aabABAb, aaaBAbAB \rangle$
$\langle a, b \mid abaBAB, aaaaBBaBaBB \rangle$
$\langle a, b \mid aaaabbb, ababaBAbAB \rangle$
$\langle a, b \mid aabaaBAB, abbbaBBBB \rangle$
$\langle a, b \mid a^4, b^3, ababaBAbAB \rangle$
$\langle a, b \mid aababAAB, abbbbaBaB \rangle$

George Havas and Colin Ramsay, Short balanced presentations of perfect groups, *Groups St Andrews 2001 in Oxford*, London Mathematical Society Lecture Note Series, Cambridge University Press (to appear).

2.4 3-generator groups

Determine as much as you can about the groups presented by:

$\{a, b, c \mid b^2c^{-1}bc,\ a^2b^{-1}ab,\ ca^{-1}b^{-1}cabc\}$
$\{a, b, c \mid b^2c^{-1}bc,\ a^2b^{-1}ab,\ b^{-1}abc^2a^{-1}c\}$
$\{a, b, c \mid b^2c^{-1}bc,\ a^2b^{-1}ab,\ ca^{-1}c^2b^{-1}ab\}$
$\{a, b, c \mid bbcbC, ccacA, abaabb\}$
$\{a, b, c \mid bbcbC, ccAca, abaabb\}$

Which ones, if any, are isomorphic?

George Havas, M.F. Newman and E.A. O'Brien, Groups of deficiency zero, em Geometric and Computational Perspectives on Infinite Groups, DIMACS Series in Discrete Mathematics and Theoretical Computer Science **25** (1996) 53–67.
George Havas, M.F. Newman and E.A. O'Brien, On the efficiency of some finite groups, *Communications in Algebra* (to appear).

The papers including Havas as an author are available at http://www.itee.uq.edu.au/∼havas/ and http://www-groups.dcs.st-and.ac.uk/∼havas/ as files:
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